10.4 DETERMINANT

A determinant is a unique value obtained from a square matrix. The value is used to compute multiplicative inverse of the square matrix.
The determinant of matrix A is denoted by det(A) or $|A|$.

1. Determinant of a 2×2 matrix

   Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| = a_{11}a_{22} - a_{12}a_{21}$

2. Determinant of a 3×3 matrix

   Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ then $|B|$

   $= b_{11}c_{11} + b_{12}c_{12} + b_{13}c_{13} = b_{21}c_{21} + b_{22}c_{22} + b_{23}c_{23} = b_{31}c_{31} + b_{32}c_{32} + b_{33}c_{33}$

   Where $c_{ij} = (-1)^{i+j} M_{ij}$, $i = 1,2,3,...$ and $j = 1,2,3,...$

   The value M_{ij}, called the **minor of element** b_{ij} is the determinant of the matrix formed after eliminating the row and column that contains the element b_{ij}. The value c_{ij} obtained from the minor of b_{ij} is called the **cofactor of the element** b_{ij}.

   For example, **minor of element** $b_{23} = \begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} = b_{11}b_{32} - b_{31}b_{12}$ and

   cofactor is $(-1)^{2+3}(b_{11}b_{32} - b_{31}b_{12}) = -b_{11}b_{32} + b_{31}b_{12}$

3. Determinant of a $n\times n$ matrix

   Let A be a $n\times n$ matrix then $|A|$ = sum of the products of each element and its cofactor in any row= $\sum_{j=1}^{n} a_{ij}c_{ij}$

   Note:
   a) The most appropriate row to use for calculating determinant is the one with the largest number of zeros
   b) If the determinant of a square matrix is zero it is a singular matrix else its non singular

Example 1

Which of the following matrices are singular?

a) $\begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 7 & 5 \\ 4 & -2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 3 & 6 \\ \frac{1}{3} & 2 & 3 \\ 2 & 5 & 11 \end{bmatrix}$ e) $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 1 \\ -5 & 4 & 2 \end{bmatrix}$

Solution

a) $\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -4-4 = -8 \neq 0 \Rightarrow$ Non singular matrix

b) $\begin{vmatrix} 1 & -3 \\ -3 & 9 \end{vmatrix} = 9-9 = 0 \Rightarrow$ Singular matrix

c) $\begin{vmatrix} 2 & 7 & 5 \\ 4 & -2 & 1 \\ 0 & 1 & 4 \end{vmatrix} = -1(-18)+4(-4-28) = -110 \neq 0 \Rightarrow$ Non singular

d) $\begin{vmatrix} 1 & 3 & 6 \\ \frac{1}{3} & 2 & 3 \\ 2 & 5 & 11 \end{vmatrix} = 1(22-15)-3\left(\frac{11}{3}-6\right)+6\left(\frac{5}{3}-4\right) = 0 \Rightarrow$ Singular matrix

e) $\begin{vmatrix} 2 & -1 & 4 \\ 0 & 3 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 3(4+20)-1(8-5) = 69 \neq 0 \Rightarrow$ Non Singular matrix

Theorem

If A and B are square matrices of the same order then;

a) $\left|A^{-1}\right| = \frac{1}{|A|}$

b) $|AB| = |A||B|$

c) $|kI_n A| = k^n |A|$ where A is $n \times n$ and k is a real number

Example 2

Given that $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 4 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 0 \\ -2 & 5 & 3 \end{bmatrix}$ find;

a) $|A|$ b) $|B|$ c) $|AB|$ d) $\left|(AB)^{-1}\right|$ e) $|2I_3 AB|$

Solution

a) $|A| = -1(2+1) + 0 - 4(1-4) = 9$

b) $|B| = -4(3-0) = -12$

c) $|AB| = |A||B| = 9 \times -12 = -108$

d) $\left|(AB)^{-1}\right| = \dfrac{1}{|A||B|} = -\dfrac{1}{108}$

e) $|2I_3AB| = 2^3|AB| = 8 \times -108 = -864$

Effect of Row Operations on Determinants

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{vmatrix} 1 & 3 & 2 \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = |A| = -5$$

$$\begin{matrix} R_3 \to R_1 \\ R_2 \\ R_1 \to R_3 \end{matrix}\begin{vmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = -1 \times |A| = 5$$

$$\begin{matrix} R_1 \\ R_2 \\ -2R_3 + R_1 \to R_3 \end{matrix}\begin{vmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & -5 & 5 \end{vmatrix} = (-2) \times (-1)|A| = -10$$

$$\begin{matrix} R_1 \\ 2R_1 + 3R_2 \to R_2 \\ R_3 \end{matrix}\begin{vmatrix} 2 & 1 & -1 \\ 1 & 5 & 1 \\ 0 & -5 & 5 \end{vmatrix} = 3 \times (-2) \times (-1) \times |A| = -30$$

Theorem

If A is a square matrix then;

- *i) interchanging any two rows (columns) has the same effect as $-1 \times |A|$,*
- *ii) replacing a row (column) by its multiple of $k \neq 0$ has the same effect as $k \times |A|$*
- *iii) replacing a row (column) by the sum of two or more rows(columns) has the same effect as $1 \times |A|$*

Example 3

Use row operations to find $\begin{vmatrix} 1 & -3 & 1 & 2 \\ 3 & 1 & 6 & 2 \\ 1 & 0 & 4 & 1 \\ 3 & 1 & -1 & 2 \end{vmatrix}$

Solution

Let $k = \begin{vmatrix} 1 & -3 & 1 & 2 \\ 3 & 1 & 6 & 2 \\ 1 & 0 & 4 & 1 \\ 3 & 1 & -1 & 2 \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$

$$\Rightarrow k = \begin{vmatrix} 1 & -3 & 1 & 2 \\ 0 & 10 & 3 & -4 \\ 0 & 3 & 3 & -1 \\ 0 & 0 & -7 & 0 \end{vmatrix} \begin{matrix} R_1 \to R_1 \\ -3R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ -R_2 + R_4 \to R_4 \end{matrix}$$

$$\Rightarrow 10 \times k = \begin{vmatrix} 1 & -3 & 1 & 2 \\ 0 & 10 & 3 & -4 \\ 0 & 0 & -21 & 2 \\ 0 & 0 & -7 & 0 \end{vmatrix} \begin{matrix} R_1 \to R_1 \\ R_2 \to R_2 \\ -3R_2 + 10R_3 \to R_3 \\ R_4 \to R_4 \end{matrix}$$

$$\Rightarrow -3 \times 10 \times k = \begin{vmatrix} 1 & -3 & 1 & 2 \\ 0 & 10 & 3 & -4 \\ 0 & 0 & -21 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} \begin{matrix} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 \\ -3R_4 + R_3 \to R_4 \end{matrix}$$

$$\therefore -3 \times 10 \times k = 1 \times 10 \times (-21) \times 2 \Rightarrow k = 14$$

Example 4

Given that $\begin{vmatrix} x & y & z \\ \frac{1}{2}p & \frac{1}{2}q & \frac{1}{2}r \\ a & b & c \end{vmatrix} = k$ find the expression for $\begin{vmatrix} p & q & r \\ a-p & b-q & c-r \\ -2x & -2y & -2z \end{vmatrix}$ in terms of k

Solution

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{vmatrix} x & y & z \\ \frac{1}{2}p & \frac{1}{2}q & \frac{1}{2}r \\ a & b & c \end{vmatrix} = k$$

$$\begin{matrix} R_2 \to R_1 \\ R_1 \to R_2 \\ R_3 \to R_3 \end{matrix} \begin{vmatrix} \frac{1}{2}p & \frac{1}{2}q & \frac{1}{2}r \\ x & y & z \\ a & b & c \end{vmatrix} = (-1) \times k = -k$$

$$\begin{matrix} 2R_1 \to R_1 \\ R_3 \to R_2 \\ R_2 \to R_3 \end{matrix} \begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} = 2\times(-1)\times(-k) = 2k$$

$$\begin{matrix} 2R_1 \to R_1 \\ -R_1 + R_2 \to R_2 \\ R_2 \to R_3 \end{matrix} \begin{vmatrix} p & q & r \\ a-p & b-q & c-r \\ -2x & -2y & -2z \end{vmatrix} = (-2)\times 2k = -4k$$

Determinants of Matrices in Triangular Form

A matrix A of order $n \times n$ is in triangular form if all elements either below or above the main diagonal are such that $a_{ij} = 0$. The two matrices shown below are in triangular form;

$$\begin{bmatrix} 11 & 0 & 0 \\ 9 & 5 & 0 \\ 8 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 7 & \frac{1}{2} & -2 \\ 0 & 15 & 6 \\ 0 & 0 & 13 \end{bmatrix}$$

Theorem:

If A is a matrix in triangular form of order $n \times n$ then $|A| = a_{11}a_{22}a_{33}...a_{nn}$ = product of the main diagonal elements

Example 5

Given that $B = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 92 & \frac{2}{51} & 0 & 0 & 0 \\ 5 & 9 & 1 & 0 & 0 \\ 11 & 2 & 4 & 3 & 0 \\ 7 & 15 & 0 & 0 & -17 \end{bmatrix}$ find $\det(B)$

Solution

$$\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 92 & \frac{2}{51} & 0 & 0 & 0 \\ 5 & 9 & 1 & 0 & 0 \\ 11 & 2 & 4 & 3 & 0 \\ 7 & 15 & 0 & 0 & -17 \end{vmatrix} = 5\times\frac{2}{51}\times 1\times 3\times -17 = -10$$

Example 6

Use row operations to find the value of $\begin{vmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 3 & 0 & 1 & 5 \\ -2 & -4 & 1 & 5 \end{vmatrix}$

Solution

Let $k = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 3 & 0 & 1 & 5 \\ -2 & -4 & 1 & 5 \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$

$$\Rightarrow k = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & -6 & 4 & -1 \\ 0 & 0 & -1 & 9 \end{vmatrix} \begin{matrix} R_1 \to R_1 \\ -R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \\ 2R_1 + R_4 \to R_4 \end{matrix}$$

$$\Rightarrow 4 \times k = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & 0 & 10 & -10 \\ 0 & 0 & -1 & 9 \end{vmatrix} \begin{matrix} R_1 \to R_1 \\ R_2 \to R_2 \\ -6R_2 + 4R_3 \to R_3 \\ R_4 \to R_4 \end{matrix}$$

$$\Rightarrow 10 \times 4 \times k = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & 0 & 10 & -10 \\ 0 & 0 & 0 & 80 \end{vmatrix} \begin{matrix} R_1 \to R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 \\ 10R_4 + R_3 \to R_4 \end{matrix}$$

$$\therefore 10 \times 4 \times k = 1 \times (-4) \times 10 \times 80$$
$$\Rightarrow k = -80$$

The Zero determinants

If A is a matrix of order $n \times n$ then $|A| = 0$ if:

a) A row or column is identical to another
b) A row or a column is a multiple of another
c) A row or a column consists of all zeros

Example 7

Given that k is a real number and $\begin{vmatrix} 1 & 2 & 3 \\ k & -5 & 0 \\ 2k & -1 & 6 \end{vmatrix} = 0$ find k

Solution

$|A| = -k(12+3) - 5(6-6k) = 0 \Rightarrow k = 2$

Determinants and Multiplicative Inverse.

Theorem

If A is a square matrix whose cofactors are defined as c_{ij}, $i \in \mathbb{R}$ and $j \in \mathbb{R}$

then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} & c_{41} \\ c_{12} & c_{22} & c_{32} & c_{42} \\ c_{13} & c_{23} & c_{33} & c_{43} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{bmatrix}$

Example 8

Given that $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 6 \\ 4 & -1 & 3 \end{bmatrix}$, find;

a) $|A|$ b) $\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$ formed from the cofactors of the elements of A

c) A^{-1}

Solution

a) $|A| = 3(15+6) - 2(3-24) + 2(-1-20) = 63$

b) $\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 21 & 21 & -21 \\ -8 & 1 & 11 \\ 2 & -16 & 13 \end{bmatrix}$

c) $A^{-1} = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 21 & 21 & -21 \\ -8 & 1 & 11 \\ 2 & -16 & 13 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{8}{63} & \frac{1}{63} & \frac{11}{63} \\ \frac{2}{63} & -\frac{16}{63} & \frac{13}{63} \end{bmatrix}$