10.3 MULTIPLICATIVE INVERSE OF A MATRIX.

If A, B and I are ***square matrices of the same order*** such that $AB = I_n$ then B is the multiplicative inverse of A and is denoted by A^{-1}.

$\Rightarrow AA^{-1} = A^{-1}A = I_n$

A matrix that has a multiplicative inverse is called an ***invertible matrix.*** A Square matrix that has no inverse is called a ***singular matrix*** while the ones with inverses are called ***non-singular***.

Multiplicative inverse of a matrix can be found by use of elementary row operations as follows;

- Merge the matrix A with an identity matrix I of the same order.
- Perform row operations until the original matrix turns into an identity matrix
- A^{-1} is the resultant matrix replacing the I after row operation is complete.

Non Singular (Invertible) Matrices

Example 1

Determine the inverse of the following matrices using row operations;

a) $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ 3 & -5 & 7 \end{bmatrix}$

d) $D = \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 3 & 2 & \frac{1}{2} & 2 \\ 3 & 2 & 1 & \frac{1}{2} \\ 1 & 1 & 0 & \frac{1}{2} \end{bmatrix}$ e) $E = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 1 & 1 & 2 & -2 \\ -2 & 3 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

Solution

a) $\left[\begin{array}{cc:cc} 2 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$

$$\begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -3R_1 + 2R_2 \to R_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 4 & -3 & 2 \end{array}\right]$$

$$\begin{array}{l} -\frac{1}{4}R_2 + R_1 \to R_1 \\ \frac{1}{4}R_2 \to R_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{4} & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

A is a non-singular matrix

b) $$\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 3 & -7 & 1 & -2 & 0 \\ 0 & 1 & -7 & 0 & -3 & 1 \end{bmatrix} \begin{array}{l} R_2 \to R_1 \\ -2R_2 + R_1 \to R_2 \\ -3R_2 + R_3 \to R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & -\frac{7}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & -14 & 1 & 7 & -3 \end{bmatrix} \begin{array}{l} R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ -3R_3 + R_2 \to R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\dfrac{2}{7} & -1 & \dfrac{6}{7} \\ 0 & 1 & 0 & \dfrac{1}{2} & \dfrac{1}{2} & -\dfrac{1}{2} \\ 0 & 0 & 1 & \dfrac{1}{14} & \dfrac{1}{2} & -\dfrac{3}{14} \end{bmatrix} \begin{array}{l} -\dfrac{2}{7}R_3 + R_1 \to R_1 \\ -\dfrac{1}{6}R_3 + R_2 \to R_2 \\ \dfrac{1}{14}R_3 \to R_3 \end{array}$$

$$B^{-1} = \begin{bmatrix} -\frac{2}{7} & -1 & \frac{6}{7} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{14} & \frac{1}{2} & -\frac{3}{14} \end{bmatrix}$$

B is a non-singular matrix

c) $$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 6 & 0 & 1 & 0 \\ 3 & -5 & 7 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{r} R_1 \to R_1 \\ -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -5 & 1 & 1 \end{array}\right] \begin{array}{r} \frac{1}{2}R_2 + R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ R_3 + R_2 \to R_3 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 5 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{5}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{r} -\frac{2}{7}R_3 + R_1 \to R_1 \\ -\frac{1}{3}R_3 + R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array}$$

$$C^{-1} = \left[\begin{array}{ccc} 5 & -\frac{1}{2} & -1 \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

C is a non-singular matrix

d)

$$\left[\begin{array}{cccccccc} 1 & 1 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & \frac{1}{2} & 2 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\left[\begin{array}{cccccccc} 1 & 1 & \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 0 & -1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{r} R_1 \\ -3R_1 + R_2 \\ -R_2 + R_3 \\ -R_1 + R_4 \end{array}$$

$$\left[\begin{array}{cccccccc} 1 & 0 & -\frac{1}{2} & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 0 & -1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1+R_2 \\ -R_2 \\ R_3 \\ R_4 \end{array}$$

$$\left[\begin{array}{cccccccc} 1 & 0 & 0 & \frac{1}{2} & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} R_1+R_3 \\ -2R_3+R_2 \\ 2R_3 \\ R_3+R_4 \end{array}$$

$$\left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & -\frac{5}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 3 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 \end{array}\right] \begin{array}{c} \frac{1}{2}R_4+R_1 \\ R_4+R_2 \\ -3R_4+R_3 \\ -R_4 \end{array}$$

$$D^{-1}=\left[\begin{array}{cccc} -\frac{5}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & 0 & -1 & 1 \\ 3 & 1 & -1 & -3 \\ 1 & 1 & -1 & -1 \end{array}\right]$$

D is a non-singular matrix

e) $$\left[\begin{array}{cccccccc} 3 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\left[\begin{array}{cccccccc} 1 & 2 & 2 & -2 & 0 & 1 & 0 & 0 \\ 0 & 7 & 5 & -5 & 0 & 2 & 1 & 0 \\ 0 & 13 & 5 & 3 & 2 & 0 & 3 & 0 \\ 0 & 7 & 5 & 1 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} R_2 \to R_1 \\ 2R_2+R_3 \to R_2 \\ 2R_1+3R_3 \to R_3 \\ 2R_4+R_3 \to R_4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & \frac{4}{7} & -\frac{4}{7} & 0 & \frac{3}{7} & -\frac{2}{7} & 0 \\ 0 & 1 & \frac{5}{7} & -\frac{5}{7} & 0 & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & -30 & 86 & 14 & -26 & 8 & 0 \\ 0 & 0 & 0 & 6 & 0 & -2 & 0 & 2 \end{bmatrix} \begin{matrix} -\frac{2}{7}R_2 + R_1 \to R_1 \\ \frac{1}{7}R_2 \to R_2 \\ -13R_2 + 7R_3 \to R_3 \\ -R_2 + R_4 \to R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{16}{15} & \frac{4}{15} & -\frac{1}{15} & -\frac{2}{15} & 0 \\ 0 & 1 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{43}{15} & -\frac{7}{5} & \frac{13}{15} & \frac{4}{15} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{matrix} \frac{2}{105}R_3 + R_1 \to R_1 \\ \frac{1}{42}R_3 + R_2 \to R_2 \\ -\frac{1}{30}R_3 \to R_3 \\ \frac{1}{6}R_4 \to R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{4}{15} & \frac{13}{45} & -\frac{2}{15} & -\frac{16}{45} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{9} & \frac{1}{3} & -\frac{4}{9} \\ 0 & 0 & 1 & 0 & -\frac{7}{15} & -\frac{4}{45} & -\frac{4}{15} & \frac{43}{45} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{matrix} -\frac{16}{15}R_4 + R_1 \to R_1 \\ -\frac{4}{3}R_4 + R_2 \to R_2 \\ \frac{43}{15}R_4 + R_3 \to R_3 \\ R_4 \to R_4 \end{matrix}$$

$$E^{-1} = \begin{bmatrix} \frac{4}{15} & \frac{13}{45} & -\frac{2}{15} & -\frac{16}{45} \\ \frac{1}{3} & \frac{1}{9} & \frac{1}{3} & -\frac{4}{9} \\ -\frac{7}{15} & -\frac{4}{45} & -\frac{4}{15} & \frac{43}{45} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

E is a non-singular matrix

<u>Singular Matrices</u>

Example 2

Use row operations to determine the inverses of the following;

a) $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ b) $Q = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 4 & 5 \\ 1 & 3 & 1 \end{bmatrix}$ c) $R = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 \\ 4 & 1 & 3 & 0 \end{bmatrix}$

Solution

a) $$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\begin{array}{r} R_1 \to R_1 \\ 2R_1 + R_2 \to R_2 \end{array}\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ has no inverse.

b) $$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & -5 & 2 & 0 & 1 & -3 \\ 0 & -5 & 2 & 0 & 1 & -2 \end{array}\right]\begin{array}{r} R_3 \to R_1 \\ -3R_3 + R_2 \to R_2 \\ -2R_3 + R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{11}{5} & 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & 1 & -\frac{2}{5} & 0 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\begin{array}{r} \frac{3}{5}R_2 + R_1 \to R_1 \\ -\frac{1}{5}R_2 \to R_2 \\ -R_2 + R_3 \to R_3 \end{array}$$

$Q = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 4 & 5 \\ 1 & 3 & 1 \end{bmatrix}$ has no inverse.

c)

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -4 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_1 \\ -2R_2 + R_1 \to R_2 \\ -3R_2 + R_3 \to R_3 \\ -4R_2 + R_4 \to R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 \\ R_2 \to R_2 \\ -R_2 + R_3 \to R_3 \\ -R_3 + R_4 \to R_4 \end{matrix}$$

$R = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 \\ 4 & 1 & 3 & 0 \end{bmatrix}$ has no inverse.

P, Q and R are singular matrices.

Theorem

If k is a scalar and A and B are square matrices then;

1. $\left(A^{-1}\right)^{-1} = A$

2. $(kA)^{-1} = \frac{1}{k} A^{-1}$

3. $(AB)^{-1} = B^{-1}A^{-1}$

Example 4

a) Given that $A^{-1} = \begin{bmatrix} 2 & 6 \\ 4 & -4 \end{bmatrix}$ find $\left(\frac{2}{3}A\right)^{-1}$

b) Given that $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 2 & 5 & -1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$ find $(AB)^{-1}$.

c) Given that $P^{-1} = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 8 & 0 \\ 8 & 4 & 12 \end{bmatrix}$ and $Q = 4P$ find Q^{-1}

Solution

a) $\left(\frac{2}{3}A\right)^{-1} = \frac{3}{2}A^{-1} = \frac{3}{2}\begin{bmatrix} 2 & 6 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 6 & -6 \end{bmatrix}$

b) $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 8 & 9 & 5 \\ 9 & 14 & 2 \\ 0 & -5 & 5 \end{bmatrix}$

c) $Q^{-1} = (4P)^{-1} = \frac{1}{4}P^{-1} = \frac{1}{4}\begin{bmatrix} 4 & -4 & 8 \\ -4 & 8 & 0 \\ 8 & 4 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

Example 5

Find an expression for A^{-1} in terms of A if A is a matrix such that $A^4 - 3I_n = 2A^3 + A^2$

Solution

$\Rightarrow A^{-1}\left(A^4 - 3I_n\right) = A^{-1}\left(2A^3 + A^2\right)$
$\Rightarrow A^3 - 3A^{-1} = 2A^2 + A$
$\Rightarrow -3A^{-1} = 2A^2 + A - A^3$
$\Rightarrow A^{-1} = \frac{1}{3}A^3 - \frac{2}{3}A^2 - \frac{1}{3}A$

System of Linear Equations and Inverse Matrix

If $AX = B$ is a matrix equation then $A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$

Example 6

Solve the system of linear equations using the inverse matrix method.

a) $\begin{array}{l} 3x - 2y = 5 \\ 4x + 3y = 1 \end{array}$ b) $\begin{array}{l} 2x + y - z = -1 \\ 6x + 4y - z = 3 \\ 4x + 2y - 3z = -5 \end{array}$

Solution

a) $\left.\begin{array}{l} 3x - 2y = 5 \\ 4x + 3y = 1 \end{array}\right\} \Leftrightarrow \begin{bmatrix} 3 & -2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 3 & -2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc:cc} 3 & -2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{3}R_1 \to R_1 \\ -4R_1+3R_2 \to R_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 17 & -4 & 3 \end{array}\right]$$

$$\begin{array}{l} R_1+\frac{2}{51}R_2 \to R_1 \\ \frac{1}{17}R_2 \to R_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{9}{51} & \frac{6}{51} \\ 0 & 1 & -\frac{4}{17} & \frac{3}{17} \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{9}{51} & \frac{6}{51} \\ -\frac{4}{17} & \frac{3}{17} \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{2}{17} \\ -\frac{4}{17} & \frac{3}{17} \end{bmatrix} =$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{3}{17} & \frac{2}{17} \\ -\frac{4}{17} & \frac{3}{17} \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The solution set is $\{(1,-1)\}$

b) $$\left.\begin{array}{r} 2x+y-z=-1 \\ 6x+4y-z=3 \\ 4x+2y-3z=-5 \end{array}\right\} \Leftrightarrow \begin{bmatrix} 2 & 1 & -1 \\ 6 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & -1 \\ 6 & 4 & -1 \\ 4 & 2 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 6 & 4 & -1 & 0 & 1 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{bmatrix} \begin{array}{r} \frac{1}{2}R_1 \to R_1 \\ -3R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \begin{array}{r} -\frac{1}{2}R_2+R_1 \to R_1 \\ R_2 \to R_2 \\ -R_3 \to R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & -7 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \begin{array}{r} \frac{3}{2}R_3+R_1 \to R_1 \\ -2R_3+R_2 \to R_2 \\ R_3 \to R_3 \end{array}$$

$$A^{-1} = \begin{bmatrix} 5 & -\frac{1}{2} & -\frac{3}{2} \\ -7 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & -\frac{1}{2} & -\frac{3}{2} \\ -7 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

The solution set is $\{(1, 0, 3)\}$

Note: The inverse matrix method only offers solution to independent systems of linear equations.