

1. Discrete Fourier Transform (DFT)

- The DTFT $X(\omega)$ is a continuous function with infinite duration and the time-domain sequence $x[n]$ is also assumed to be infinite long.
- In practice, the continuous function of $X(\omega)$ is very difficult to be handled by the digital computer. In addition, most of DSP applications are expected to process a finite-duration input sequence (say, N elements). Therefore, we need a transform domain representation for finite extent sequences and the representation should also be a finite extent sequence $X(k)$ for $k = 0, 1, \dots, N-1$.
- The discrete Fourier transform (DFT) is defined to satisfy these two requirements.

1.1 Discrete Fourier Series (DFS) – Discrete-Time version of Fourier Series

A. Fourier Series: A Frequency Analysis Technique for Continuous-Time Periodic Signals

A continuous-time function $x(t)$ is said to be periodic with fundamental period T if

$$x(t) = x(t + T), \quad t \in (-\infty, \infty)$$

Every periodic function can be expanded into a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in (-\infty, \infty)$$

where
$$c_k = \frac{\omega_0}{2\pi} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

and $\omega_0 = 2\pi/T$ is called the fundamental frequency.

Since $e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$, another form of the Fourier series is given by

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t), \quad t \in (-\infty, \infty)$$

where

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt,$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(k\omega_0 t) dt,$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(k\omega_0 t) dt.$$

B. Representation of Periodic Sequences: The Discrete Fourier Series

Consider a sequence $\tilde{x}[n]$ that is periodic with period N , so that $\tilde{x}[n] = \tilde{x}[n + rN]$, for any integer values of n and r . As with continuous-time periodic signals, such a sequence can be represented by a Fourier series corresponding to a sum of harmonically related complex exponential sequences, i.e., complex exponentials with frequencies that are integer multiples of the fundamental frequency $(2\pi/N)$ associated with the periodic sequence $\tilde{x}[n]$. These periodic complex exponentials are of the form

$$e_k[n] = e^{j(2\pi/N)kn} = e_k[n + rN],$$

where k is an integer, and the Fourier series representation then has the form

$$\tilde{x}[n] = \frac{1}{N} \sum_k \tilde{X}[k] \cdot e^{j(2\pi/N)kn}.$$

The Fourier series representation of a continuous-time periodic signal generally requires infinitely many harmonically related complex exponentials, whereas the Fourier series for any discrete-time signal with period N requires only N harmonically related complex exponentials.

Since $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, and, in general,

$$e_{k+lN}[n] = e^{j(2\pi/N)(k+lN)n} = e^{j(2\pi/N)kn} e^{j2\pi ln} = e^{j(2\pi/N)kn} = e_k[n]$$

where l is an integer. Thus, the Fourier series representation of a periodic sequence $\tilde{x}[n]$ need contain only N of these complex exponentials, and hence, it has the form

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \cdot e^{j(2\pi/N)kn} \quad (1)$$

The Fourier series coefficients $\tilde{X}[k]$ are obtained from $\tilde{x}[n]$ by the relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] \cdot e^{-j(2\pi/N)kn} \quad (2)$$

Note that the sequence $\tilde{X}[k]$ is discrete and periodic with period N . In addition, theoretically

the duration is infinite extent.

Equations (1) and (2) together are referred to as the discrete Fourier series (DFS) representation of a periodic sequence.

Example 1:

A periodic sequence is given by

$$\{\tilde{x}[n]\} = \{\dots, 0, 1, 3, 5, 0, 1, 3, 5, 0, 1, 3, 5, \dots\}$$

↑

[Solution]

The fundamental period of $\{\tilde{x}[n]\}$ is $N = 4$, and

$$\tilde{X}[k] = \sum_{n=0}^3 \tilde{x}[n] \cdot e^{-j(2\pi/4)kn}$$

$$\text{Now } \tilde{X}[0] = \sum_{n=0}^3 \tilde{x}[n] \cdot e^{-j(2\pi/4)0n} = \sum_{n=0}^3 \tilde{x}[n] = 0 + 1 + 3 + 5 = 9.$$

$$\tilde{X}[1] = \sum_{n=0}^3 \tilde{x}[n] \cdot e^{-j(2\pi/4)n} = \sum_{n=0}^3 \tilde{x}[n](-j)^n = -3 + 4j.$$

$$\tilde{X}[2] = \sum_{n=0}^3 \tilde{x}[n] \cdot e^{-j(2\pi/4)2n} = \sum_{n=0}^3 \tilde{x}[n](-j)^{2n} = -3.$$

$$\tilde{X}[3] = \sum_{n=0}^3 \tilde{x}[n] \cdot e^{-j(2\pi/4)3n} = \sum_{n=0}^3 \tilde{x}[n](-j)^{3n} = -3 - 4j.$$

C. Properties of Discrete Fourier Series

Linearity : $A\tilde{x}[n] + B\tilde{y}[n] \leftrightarrow A\tilde{X}[k] + B\tilde{Y}[k]$

Time Shift : $\tilde{x}[n - m] \leftrightarrow e^{-j(2\pi/N)km} \tilde{X}[k]$

Frequency Shift : $e^{j(2\pi/N)nl} \tilde{x}[n] \leftrightarrow \tilde{X}[k - l]$

Periodic Convolution: $\tilde{x}_1[n] \otimes \tilde{x}_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n - m] \leftrightarrow \tilde{X}_1[k] \tilde{X}_2[k]$

Parseval's Theorem : $\sum_{n=0}^{N-1} |\tilde{x}[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\tilde{X}[k]|^2$

D. Relationship between DFS and DTFT

Let one period of $\tilde{x}[n]$ form a non-periodic sequence $x[n]$ and recall that the DTFT of a sequence $x[n]$ exists if it is absolutely summable, i.e., $\sum |x[n]| < \infty$. Its DTFT is given by

$$X(\omega) = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n] \cdot e^{-j\omega n}$$

Comparing it with the DFS representation, then

$$\tilde{X}[k] = X(\omega) \Big|_{\omega=k(2\pi/N)}$$

This equation shows that DFS is a frequency domain sampling of the DTFT. The DFS is obtainable by evenly sampling the DTFT at intervals of $\omega_1 = 2\pi/N$. In addition, one period of the DFS is an efficient frequency domain representation of a finite extent sequence. It is because one period of the DFS has sufficient information to reconstruct the finite extent sequence. Based on this idea, the DFT is defined as one period of the DFS for the periodic version of a finite extent sequence.

Example 2:

A sequence $\{x[n]\} = \{0, 1, 3, 5\}$, compute its DTFT and by sampling the DTFT at $k\omega_1 = (2\pi/4)k, k = 0, 1, 2, 3$, show that it is equal to its DFS $\tilde{X}[k]$.

[Solution]

The DTFT is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} = e^{-j\omega} + 3e^{-2j\omega} + 5e^{-j3\omega}.$$

Sampling at $k\omega_1 = (2\pi/4)k, k = 0, 1, 2, 3$, we have

$$\begin{aligned} X(e^{j0}) &= \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} = 9 = \tilde{X}[0]. \\ X(e^{j2\pi/4}) &= e^{-j2\pi/4} + 3e^{-j4\pi/4} + 5e^{-j6\pi/4} = \tilde{X}[1] \\ X(e^{j4\pi/4}) &= e^{-j4\pi/4} + 3e^{-j8\pi/4} + 5e^{-j12\pi/4} = \tilde{X}[2] \\ X(e^{j6\pi/4}) &= e^{-j6\pi/4} + 3e^{-j12\pi/4} + 5e^{-j18\pi/4} = \tilde{X}[3] \end{aligned}$$

as expected.

1.2 Discrete Fourier Transform

A finite-length sequence $x[n]$ with $0 \leq n \leq N-1$ non-zero samples can be replicated to form a periodic sequence $\tilde{x}[n]$ with

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n + kN]$$

The periodic sequence $\tilde{x}[n]$ has a discrete and periodic frequency domain representation of the DFS $\tilde{X}[k]$, which can be truncated by a rectangular window function $R_N[n]$ to form a finite extent sequence as

$$X[k] = \tilde{X}[k] \cdot R_N[k]$$

where
$$R_N[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$R_N[n]$ is a window function that truncates a periodic sequence to a finite extent sequence with one period of samples. The frequency domain representation $X[k]$ is called the discrete Fourier transform (DFT) of the sequence $x[n]$, which is a finite-length sequence and has sufficient information to reconstruct the sequence $x[n]$. The relationship among these sequences can be summarized as follows:

$$x[n] \leftrightarrow \tilde{x}[n] \leftrightarrow \tilde{X}[k] \leftrightarrow X[k]$$

and
$$x[n] = \tilde{x}[n] \cdot R_N[n], \quad \tilde{x}[n] = \sum_{i=-\infty}^{\infty} x[n + iN]$$

$$X[k] = \tilde{X}[k] \cdot R_N[k], \quad \tilde{X}[k] = \sum_{i=-\infty}^{\infty} X[k + iN].$$

The DFT pair is defined as

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] \cdot e^{-j(2\pi/N)kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{j(2\pi/N)kn}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

1.3 Relationship between DFT and DTFT

Basically, the DFT is a frequency domain sampling on one period of the DTFT, i.e.,

$$X[k] = X(\omega) \Big|_{\omega=k(2\pi/N)} \quad k=0, 1, \dots, N-1$$

When using DFT to compute an approximation to the DTFT of an aperiodic sequence, it is usually necessary to modify the original sequence by windowing and/or zero-padding.

1.4 Properties of DFT

Properties of the DFT are very much similar to other Fourier Transform or Fourier Series.

Linearity

The DFT and inverse DFT are linear operators:

$$Ax[n] + By[n] \leftrightarrow AX[k] + BX[k]$$

Time Shift

A delay of m samples in the time domain corresponds to multiplying the transform by $e^{-j2\pi km/N}$

$$x[n - m] \leftrightarrow X[k]e^{-j2\pi km/N}$$

Since the sequence is assumed to be periodic, the delayed sequence still consists of the original N values which are shifted in a circular fashion. e.g. $\{x[n - 1]\} = \{x_3, x_0, x_1, x_2\}$.

Circular Convolution

Convolving two time sequences is equivalent to multiplying their DFTs.

$$x[n] \overset{\circ}{\otimes} y[n] = [\tilde{x}[n] \overset{\circ}{\otimes} \tilde{y}[n]] \cdot R_N[n] \leftrightarrow X[k] \cdot Y[k]$$

Note that since the time sequences are assumed to be periodic with period N , the convolution is “circular”.

Modulation

Conversely, multiplying two sequences in time domain is equivalent to convolving their DFTs.

$$x[n]y[n] \leftrightarrow X[k] \overset{\circ}{\otimes} Y[k]$$

Parseval's Theorem

The total power in the time and frequency domain is equal:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Example 3:

Let $\{x_1[n]\} = \{1, 2, 2\}$ and $\{x_2[n]\} = \{1, 2, 3, 4\}$. Compute the 4-point circular convolution $x_1[n] \overset{\circ}{\otimes} x_2[n]$.

[Solution]

Note that the sequence $x_1[n]$ is a 3-point sequence while $x_2[n]$ is a 4-point one. Hence we will have to pad one '0' to $x_1[n]$ to make it a 4-point sequence before circular convolution.

1. *Time-domain approach:*

The 4-point circular convolution is given by

$$x_1[n] \overset{\circ}{\otimes} x_2[n] = \sum_{m=0}^3 x_1[m] x_2[(n-m)_4]$$

for $n = 0$

$$\begin{aligned} \sum_{m=0}^3 x_1[m] x_2[(-m)_4] &= \sum_{m=0}^3 \{1, 2, 2, 0\} \{1, 4, 3, 2\} \\ &= \sum_{m=0}^3 \{1, 8, 6, 0\} = 15. \end{aligned}$$

for $n = 1$

$$\begin{aligned} \sum_{m=0}^3 x_1[m] x_2[(1-m)_4] &= \sum_{m=0}^3 \{1, 2, 2, 0\} \{2, 1, 4, 3\} \\ &= \sum_{m=0}^3 \{2, 2, 8, 0\} = 12. \end{aligned}$$

for $n = 2$

$$\begin{aligned} \sum_{m=0}^3 x_1[m] x_2[(2-m)_4] &= \sum_{m=0}^3 \{1, 2, 2, 0\} \{3, 2, 1, 4\} \\ &= \sum_{m=0}^3 \{3, 4, 2, 0\} = 9. \end{aligned}$$

for $n = 3$

$$\begin{aligned} \sum_{m=0}^3 x_1[m] x_2[(3-m)_4] &= \sum_{m=0}^3 \{1, 2, 2, 0\} \{4, 3, 2, 1\} \\ &= \sum_{m=0}^3 \{4, 6, 4, 0\} = 14. \end{aligned}$$

$$\Rightarrow x_1[n] \overset{\circ}{\otimes} x_2[n] = \{15, 12, 9, 14\}.$$

2. *Frequency-domain approach:*

We first compute the 4-point DFTs of $x_1[n]$ and $x_2[n]$, multiply them sample by sample, and then obtain the IDFT to estimate the circular convolution.

The DFT of $x_1[n]$:

$$X_1[k] = \sum_{n=0}^3 x_1[n] e^{-j(2\pi/4)kn}$$

$$X_1[0] = \sum_{n=0}^3 x_1[n] e^{-j(2\pi/4)0n} = 5.$$

$$\begin{aligned} X_1[1] &= \sum_{n=0}^3 x_1[n] e^{-j(2\pi/4)n} = 1 + 2 \cdot e^{-j\pi/2} + 2 \cdot e^{-j\pi} \\ &= 1 - 2 \cdot j - 2 = -1 - 2j. \end{aligned}$$

$$X_1[2] = \sum_{n=0}^3 x_1[n] e^{-j(4\pi/4)n} = 1.$$

$$X_1[3] = \sum_{n=0}^3 x_1[n] e^{-j(6\pi/4)n} = -1 + 2j.$$

$$\Rightarrow X_1[k] = \{5, -1 - 2j, 1, -1 + 2j\}$$

The DFT of $x_2[n]$:

Similarly we can obtain $X_2[k] = \{10, -2 + 2j, -2, -2 - 2j\}$.

$$\Rightarrow X_1[k] \cdot X_2[k] = \{50, 6 + 2j, -2, 6 - 2j\}.$$

The IDFT of $X_1[k] X_2[k]$:

$$x_1[n] \overset{\circ}{\otimes} x_2[n] = \frac{1}{4} \sum_{k=0}^3 X[k] \cdot e^{j(2\pi/4)kn}$$

$n=0$:

$$\frac{1}{4} \sum_{k=0}^3 X[k] \cdot e^{j(2\pi/4)k0} = 15.$$

$n=1$:

$$\begin{aligned} &\frac{1}{4} \sum_{k=0}^3 X[k] \cdot e^{j(2\pi/4)k} \\ &= \frac{1}{4} [50 + (6 + 2j) \cdot e^{j(\pi/2)} - 2 \cdot e^{j\pi} + (6 - 2j) \cdot e^{j(6\pi/4)}] \\ &= 12. \end{aligned}$$

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$$\Rightarrow x_1[n] \overset{\circ}{\otimes} x_2[n] = \{15, 12, 9, 14\}$$

which is exactly the same as the result obtained via the time-domain approach

Summary

DTFT for $x[n]$:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega.$$

DFS for a periodic

sequence $\tilde{x}[n]$:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] \cdot e^{-j(2\pi/N)kn}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \cdot e^{j(2\pi/N)kn}, \quad n = 0, \pm 1, \pm 2, \dots$$

DFT for a finite length

sequence $x[n]$ via

its extended periodic

sequence $\tilde{x}[n]$

$$x[n] \leftrightarrow \tilde{x}[n] \leftrightarrow \tilde{X}[k] \leftrightarrow X[k]$$

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] \cdot e^{-j(2\pi/N)kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{j(2\pi/N)kn}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$